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A BY-PRODUCT PRODUCTION PROCESS WITH AN ALTERNATIVE FOR A TWO-P--ETC(U)
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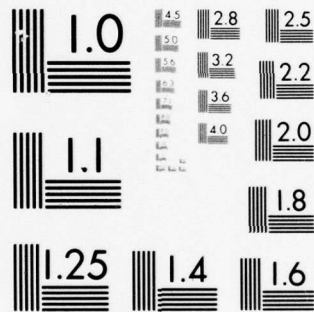
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A By-Product Production System

With an Alternative[†]

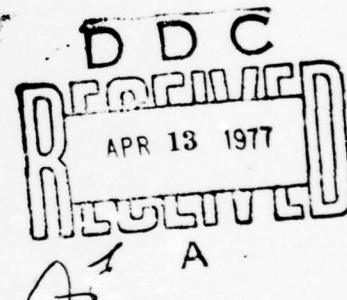
by

Bryan L. Deuermeyer*

William P. Pierskalla**

Technical Report No. 4 ✓

January 1977

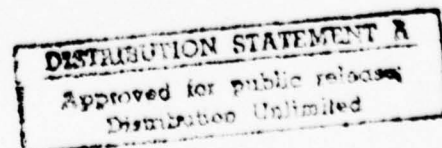


† Research partially supported by GK40034 from the National Science Foundation and prepared under Office of Naval Research Contract N00014-75-C-0797 (Task NRO42-322), W. P. Pierskalla, Project Director.

° Presented by invitation at the Fall ORSA/TIMS National Meeting in Miami Beach, Florida on November 5, 1976.

* Krannert Graduate School of Management
Purdue University
West Lafayette, Indiana 47907

** Department of Industrial Engineering and Management Sciences
Technological Institute
Northwestern University
Evanston, Illinois 60201



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Abstract

This paper considers the optimal control of a production system which is composed of two distinct production processes, types A and B, that produce two different products, I and II, having independent non-negative random demands. Production type A produces both products related by a fixed set of production coefficients. Type B can only be used to produce product II. Each period, the optimal production level of each type must be determined. The criterion is the minimum expected discounted total cost. Results show that the decision space is partitioned into four distinct regions, whose boundaries are characterized in terms of a particular sequence of points which are independent of the on-hand stock levels.

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Introduction

Although the theory of single product inventories initiated by [1, 2] and [3] is extensive, the same is not true for multiproduct inventories. The numerous papers that have appeared constitute a fraction of the myriad of possibilities that arise in complex real inventory systems. One possibility not yet considered corresponds to the problem of coordinating different types of production processes used to manufacture several stock items, when the demands are stochastic. This paper considers an inventory-production system such as the one depicted in Figure 1, having a by-product process and a single item process that together produce two products. Such a system occurs naturally (although often in much more complicated forms) in such settings as steel mills, tire manufacturing plants, chemical manufacturing plants, etc.. For specificity, it is assumed that the two production processes are labelled as types A and B while the stock items are called products 1 and 2. It is assumed that type B is designed to produce only product 2 and is called a single item process. Type A is assumed to be a generalized by-product process; a unit of production yields η_j (>0) of product j , $j=1, 2$. It is not required that $\eta_1 + \eta_2 = 1$. The parameters η_j are called production coefficients and they are fixed characteristics of type A and cannot be changed. In this context, type B is the alternative method of producing product 2. The system is of the periodic review type; i.e., at the beginning of each period (over a finite horizon) the stock levels are reviewed and the production levels of types A and B are determined.

In reviewing the literature one sees that a variety of papers have

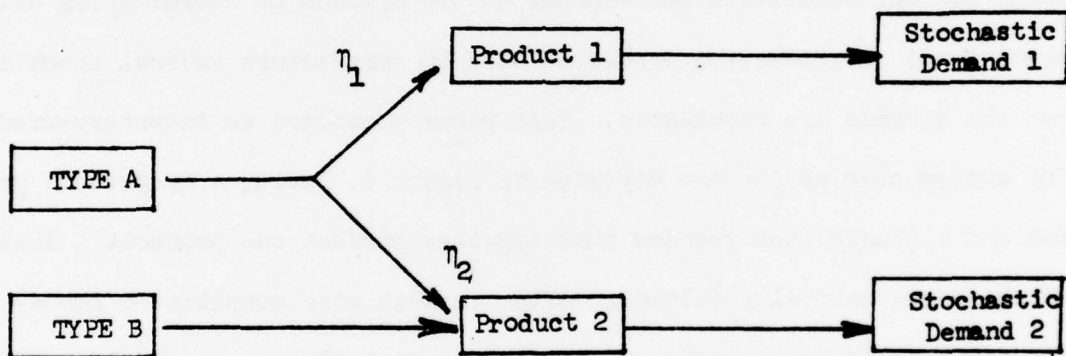


Figure 1

been written concerning the control of inventories for several products. The excellent survey paper written by Veinott [16] reviews the literature through 1965. Among the topics discussed by him are: substitute products, multilocation systems, ordering and repair and capital accumulation and production. Veinott [15] developed a very general multiproduct model that allowed for several demand categories. Evans [6] considered the control of a by-product production system using two different production cost structures: under lost sales he used a linear joint production cost and under complete backlogging he was able to generalize K-convexity to allow for a fixed set up cost. Evans [5] considered a multiproduct system when there are limited resources. Shah [13] explicitly treated different types of substitutability between products. Johnson [10] presented an infinite horizon model with fixed set up costs for an inventory system consisting of several different products. Silver [14] and Goyal [8, 9] considered the problem of joint replenishment when demands are known. Most of these papers dealt with multiproduct inventories when the stock items are related in some fashion (resource limitations, by-products, etc.). Apparently, the problem of coordinating production processes has not yet been considered, within the context of inventory theory.

The intent of this paper is to develop a dynamic model of the system described by Figure 1 so optimal production-inventory policies can be derived for controlling the system over any finite planning horizon. In the next section the necessary assumptions are stated and the appropriate notation is developed so that a precise mathematical statement of the model can be made. Section 3 contains the central result of the paper

along with other technically oriented results. The proofs of many of the results are found in the Appendix.

2. Notation, Assumptions, and Model Formulation

The following definitions, assumptions, and conventions serve to state the model mathematically.

a) In discussing the model, n will be the period index, with periods numbered backwards and with N the horizon.

b) y and z denote arbitrary production levels for types A and B, while x and w denote the stock levels at the beginning of a period (prior to production and demand).

c) The demands for products 1 and 2 in period n , D_1^n and D_2^n respectively, are independent nonnegative random variables with continuous densities $f_j(\cdot)$, $j = 1, 2$. Furthermore, the demands $\tilde{D}^n = (D_1^n, D_2^n)$ are independent and identically distributed.

d) If φ is a differentiable scalar function, the convention $\varphi'(t) = \frac{d\varphi(t)}{dt}$ is used. If $\varphi: \mathbb{R}^n$ is twice differentiable, we use

$$\varphi^{(1)}(\tilde{t}) = \frac{\partial}{\partial t_i} \varphi(\tilde{t}) \quad i = 1, 2$$

and

$$\varphi^{(1,j)}(\tilde{t}) = \frac{\partial^2 \varphi(\tilde{t})}{\partial t_i \partial t_j} \quad i, j = 1, 2.$$

e) It is assumed that a level of $y \geq 0$ costs cy for type A implying that all costs are aggregated into this linear function, where $c \geq 0$. A level of z costs kz for type B, where $k > 0$. Although it is somewhat re-

restrictive to omit setup costs, there are a variety of systems such as petroleum cracking plants where changing from one production process to another is inexpensive. However, more general cost structures are to be considered in a later article.

f) The function $L(a,b)$ represents the joint expected holding and shortage cost given a and b of products 1 and 2, respectively, on hand after production and prior to demand. It is assumed that L is strictly convex and is continuously differentiable. In addition, we assume

$$(i) \quad L(a,b) \rightarrow +\infty$$

$$(ii) \quad \eta_1^{-1}[c - \eta_2 k] a + kb + L(a,b) \rightarrow +\infty$$

whenever $|a| \rightarrow +\infty$ and $|b| \rightarrow +\infty$. The quantity $\eta_2 k - c$ represents the difference in cost resulting from producing product 2 via type B rather than A. If this cost is positive, then it is not difficult to show that product 1 is too expensive and will be produced only when type B is not used. We assume $\eta_2 k - c$ is negative (i.e., $c - \eta_2 k > 0$). The cost $\eta_1^{-1}[c - \eta_2 k]$ represents the marginal cost of producing product 1.

g) The discount cost is denoted by α and it is assumed that $0 \leq \alpha \leq 1$.

The joint cost c in (e) is similar to the one used by Evans [6], and is a convenient means of expressing the costs of operating type A. The linear cost k for type B is analogous to the linear cost used by Arrow, Karlin and Scarf [3]. The function L defined in (f) is by now standard in inventory theory, and the strict convexity is not too restrictive. The remainder of assumption (f) is used to establish the finiteness of the optimal policy in period 1 ($n=1$) and is analogous to the standard $p'(0) > c$

assumption found in the literature.

The appropriate dynamic functional equation in period n is

$$C_n(x, w) = \inf_{\substack{y \geq 0 \\ z \geq 0}} B_n(y, z, x, w)$$

where

$$\begin{aligned} B_n(y, z, x, w) &= cy + kz + L(\eta_1 y + x, z + \eta_2 y + w) + \\ &\quad \alpha \int_0^\infty \int_0^\infty C_{n-1}(\eta_1 y + x - t, z + \eta_2 y + w - u) f_1(t) f_2(u) dt du \\ &= cy + kz + J_n(\eta_1 y + x, z + \eta_2 y + w) \end{aligned}$$

where

$$J_n(x, w) = L(x, w) + \alpha \int_0^\infty \int_0^\infty C_{n-1}(x - t, w - u) f_1(t) f_2(u) dt du, \text{ for } n = 1, 2, \dots, N.$$

J_n simply represents the expected value part of B_n and is used for concise notation. The initial condition $C_0(x, w) \equiv 0$ implies that $J_n(x, w) = L(x, w)$ for $n = 1$. $B_n(y, z, x, w)$ is the total expected discounted cost starting with (x, w) and following policy (y, z) while $C_n(x, w)$ is the optimal return function, from period n to the horizon. The result of this stagewise optimization problem is to characterize the optimal production policy

$$(\tilde{y}^*(x, w), \tilde{z}^*(x, w)) = \{(y_n^*(x, w), z_n^*(x, w))\}_1^N.$$

One minor departure from standard inventory analysis made for notational convenience is that the production levels are used directly in this model rather than the target stock levels.

The central result of this article is that the decision space and optimal policy are characterized in terms of a point (\bar{x}_n, \bar{w}_n) and monotone func-

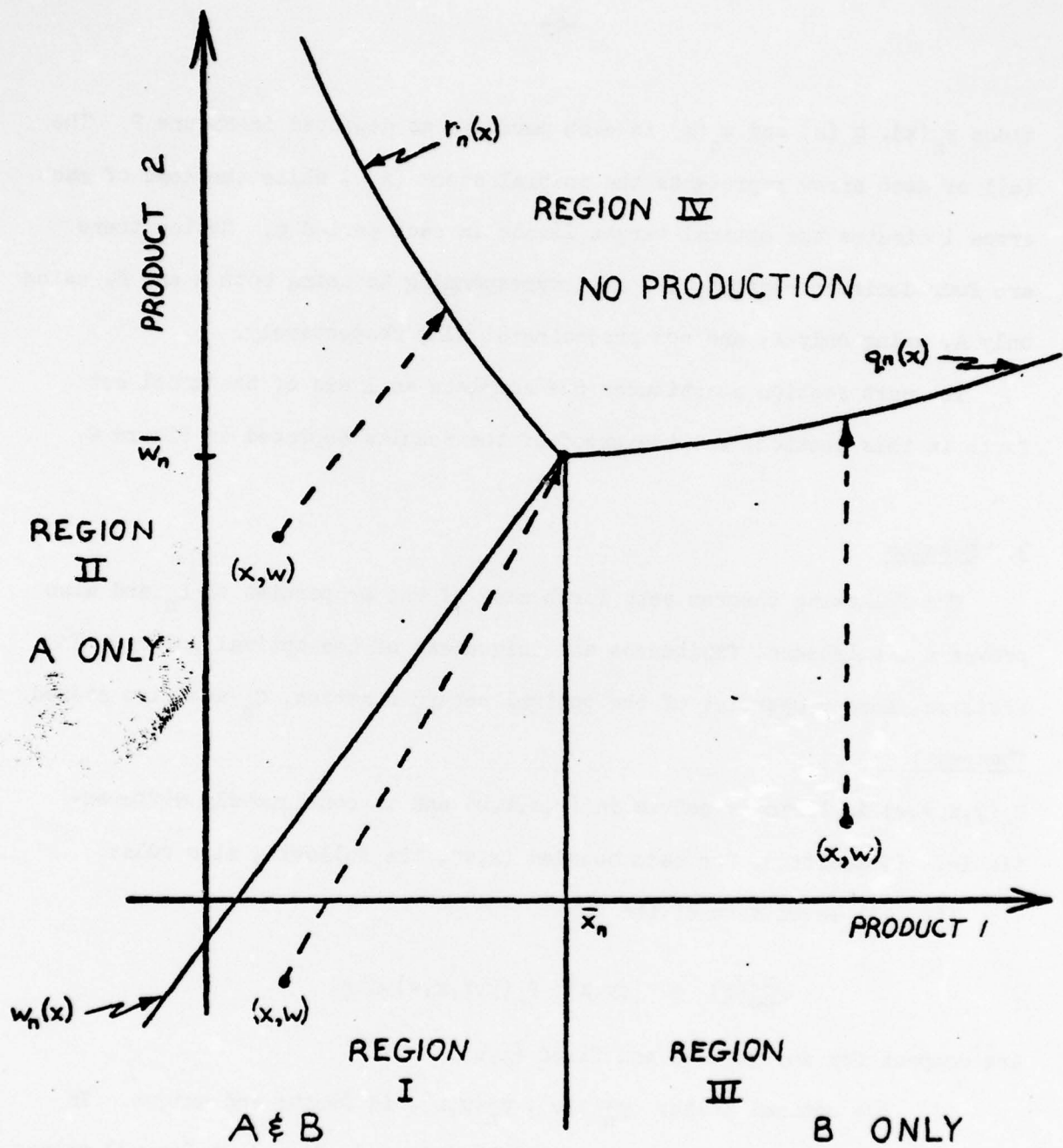


Figure 2

Characterization of Decision Plane for $n \geq 1$.

(x, w) is initial position, arrowhead
marks resultant after ordering.

tions $r_n(x)$, $q_n(x)$ and $w_n(x)$ in each period n as depicted in Figure 2. The tail of each arrow represents the initial stock (x,w) while the head of each arrow indicates the optimal target levels in each period n . Notice there are four decision regions, I - IV, corresponding to using both A and B, using only A, using only B, and not producing at all, respectively.

The next section constitutes the complete analysis of the model set forth in this section, and the proof of the results depicted in Figure 2.

3. Results

The following theorem sets forth many of the properties of B_n and also proves the existence, finiteness and uniqueness of the optimal policy. In addition, some properties of the optimal return function, C_n are also stated.

Theorem 1

$B_n(y,z,x,w)$ is strictly convex in (y,z,x,w) and is continuously differentiable. In addition, for each bounded (x,w) , the following also hold:

- (1) All level sets of the form

$$\xi_{xw}^n(\zeta) = \{(y,z); B_n(y,z,x,w) \leq \zeta\}$$

are compact for any $\zeta \in [0, \infty)$ and fixed (x,w) .

- (2) The optimal policy $(y_n^*(x,w), z_n^*(x,w))$ is finite and unique. In addition, the unique unconstrained global minimum $(y_n^u(x,w), z_n^u(x,w))$ solves

$$B_n^{(j)}(y_n^u(x,w), z_n^u(x,w), x,w) = 0, \quad j = 1, 2.$$

- (3) The two systems

$$B_n^{(1)}(\bar{y}_n(x,w), 0, x,w) = 0$$

and $B_n^{(2)}(0, \bar{z}_n(x,w), x,w) = 0$

where $B_n(y, z, x, w)$ reaches its minimum first when $y=0$ and then when $z=0$ have unique finite solutions $\bar{y}_n(x, w)$ and $\bar{z}_n(x, w)$, respectively.

(4) $C_n(x, w)$ is convex and continuous.

Since the optimal policy is constrained to be nonnegative, one can see there are four possibilities: $y_n^*(x, w) > 0, z_n^*(x, w) > 0$; $y_n^*(x, w) > 0, z_n^*(x, w) = 0$; $y_n^*(x, w) = 0, z_n^*(x, w) > 0$; and $y_n^*(x, w) = z_n^*(x, w) = 0$. The next theorem demonstrates that all four combinations can indeed occur, and specifies the conditions used to define the optimal policy in each case.

Theorem 2

The optimal policy is characterized by the following four regions.

(a) Region I (Both A and B are used):

$$y_n^*(x, w) > 0 \quad \text{and} \quad z_n^*(x, w) > 0$$

can occur iff (x, w) satisfies

$$y_n^u(x, w) > 0 \quad \text{and} \quad z_n^u(x, w) > 0 ;$$

In which case $(y_n^*(x, w), z_n^*(x, w)) = (y_n^u(x, w), z_n^u(x, w))$.

(b) Region II (Only A is used):

$$y_n^*(x, w) > 0 \quad \text{and} \quad z_n^*(x, w) = 0$$

can occur iff (x, w) satisfies

$$y_n^u(x, w) > 0, \bar{y}_n(x, w) > 0 \quad \text{and} \quad z_n^u(x, w) \leq 0;$$

In which case $(y_n^*(x, w), z_n^*(x, w)) = (\bar{y}_n(x, w), 0)$.

(c) Region III (Only B is used):

$$y_n^*(x,w) = 0 \text{ and } z_n^*(x,w) > 0$$

can occur iff (x,w) satisfies

$$y_n^u(x,w) \leq 0 \text{ and } z_n^u(x,w) > 0, \bar{z}_n(x,w) > 0;$$

in which case $(y_n^*(x,w), z_n^*(x,w)) = (0, z_n^u(x,w))$.

(d) Region IV (No production):

$$y_n^*(x,w) = 0 \text{ and } z_n^*(x,w) = 0$$

can occur iff (x,w) satisfies one of the following:

$$(i) (y_n^u(x,w), z_n^u(x,w)) \leq 0;$$

$$(ii) y_n^u(x,w) > 0, z_n^u(x,w) \leq 0 \text{ but } \bar{y}_n(x,w) \leq 0;$$

$$(iii) y_n^u(x,w) \leq 0, z_n^u(x,w) > 0 \text{ but } \bar{z}_n(x,w) \leq 0.$$

Notice that all of the events involving (x,w) in (a) - (d) are mutually exclusive and exhaustive.

The next theorem presents some of the properties of the optimal policy as (x,w) varies.

Theorem 3

1. $B_n(y,z,x,w)$ is continuously twice differentiable. In addition, the optimal policy is piecewise differentiable. Furthermore, the partial derivatives of the optimal policy satisfies the following relations:

(a) In region I:

$$y_n^{*(1)}(x,w) = -\eta_1^{-1}$$

$$z_n^{*(1)}(x,w) = \eta_2 \eta_1^{-1}$$

$$y_n^{*(2)}(x,w) = 0$$

$$z_n^{*(2)}(x,w) = -1.$$

(b) In region II:

$$-1 < \eta_1 y_n^{*(j)}(x, w) < 0$$

$$z_n^{*(j)}(x, w) = 0 \quad j=1, 2.$$

(c) In region III:

$$y_n^{*(j)}(x, w) = 0 \quad j=1, 2.$$

$$z_n^{*(1)}(x, w) > 0 \quad \text{and} \quad z_n^{*(2)}(x, w) = -1.$$

(d) In region IV:

$$y_n^{*(j)}(x, w) = 0$$

$$z_n^{*(j)}(x, w) = 0, \quad j=1, 2.$$

2. $c_n^{(j)}(x, w)$ has piecewise continuous derivatives for $j=1, 2$.

Furthermore,

$$\eta_1 c_n^{(j,1)}(x, w) + \eta_2 c_n^{(j,2)}(x, w) \geq 0, \quad j=1, 2$$

and

$$c_n^{(1,2)}(x, w) = c_n^{(2,1)}(x, w) \leq 0.$$

Theorem 3 shows that the partial derivatives of the optimal policy are piecewise continuous, with the points of discontinuity occurring on the boundaries between regions. Part (a) indicates that in region I, $y_n^*(x, w)$ is linear and decreasing in x . In the same region, $z_n^*(x, w)$ is linear by decreasing in both x and w . These properties are exploited in the main result, Theorem 4. The results in parts (b), (c), and (d) are intuitive, except for the unexpected property that $z_n^*(x, w)$ is nondecreasing in x in region III (part (c)).

As mentioned above, the linearities in region I are the key to the Theorem 4. The following points and functions are needed for the Theorem and some of their properties are given in Lemma 3.

Let the point (\bar{x}_n, \bar{w}_n) solve

$$B_n^{(j)}(0, 0, \bar{x}_n, \bar{w}_n) = 0 \quad j=1, 2.$$

Furthermore, define functions $r_n(x)$, $q_n(x)$, and $w_n(x)$ pointwise in the following manner:

$$B_n^{(1)}(0, 0, x, r_n(x)) = 0, \quad x \leq \bar{x}_n$$

$$B_n^{(2)}(0, 0, x, q_n(x)) = 0, \quad x \geq \bar{x}_n$$

and

$$w_n(x) \triangleq \bar{w}_n - \eta_2 \eta_1^{-1} [\bar{x}_n - x].$$

Then, the following lemma can be stated.

Lemma 3.

(1) (\bar{x}_n, \bar{w}_n) is finite and unique.

(2) $r_n(x)$ that solves

$$B_n^{(1)}(0, 0, x, r_n(x)) = 0, \quad x \leq \bar{x}_n$$

is finite and unique.

(3) $q_n(x)$ solving

$$B_n^{(2)}(0, 0, x, q_n(x)) = 0, \quad x \geq \bar{x}_n$$

is unique and finite.

(4) $w_n(x) \triangleq \bar{w}_n - \eta_2 \eta_1^{-1} [\bar{x}_n - x]$ is well defined.

$$(5) \quad (a) \quad r_n'(x) < 0, \quad x \leq \bar{x}_n$$

$$\text{and} \quad r_n(\bar{x}_n) = \bar{w}_n.$$

$$(b) \quad 0 < q_n'(x) < 1, \quad x > \bar{x}_n$$

$$\text{and} \quad q_n(\bar{x}_n) = \bar{w}_n$$

$$(c) \quad w_n'(x) = \eta_2 \eta_1^{-1}$$

$$\text{and} \quad w_n(x) = \bar{w}_n.$$

The characterization of the ordering regions can now be stated in Theorem 4.

Theorem 4

The optimal policy and the four decision regions are uniquely characterized as follows.

(a) Region I (Both A and B are used):

If $x < \bar{x}_n$ and $w < w_n(x)$, then

$$y_n^*(x, w) = \eta_1^{-1} [\bar{x}_n - x] \quad (> 0)$$

$$z_n^*(x, w) = w_n(x) - w \quad (> 0), \text{ and conversely.}$$

(b) Region II (Only A is used):

If $x < \bar{x}_n$ and $w_n(x) \leq w < r_n(x)$, then

$$y_n^*(x, w) > 0 \quad \text{and} \quad z_n^*(x, w) = 0$$

and

$$r_n(x + \eta_1 y_n^*(x, w)) = w + \eta_2 y_n^*(x, w), \text{ and conversely.}$$

(c) Region III (Only B is used):

If $x \geq \bar{x}_n$ and $w < q_n(x)$, then

$$y_n^*(x, w) = 0$$

and

$$z_n^*(x, w) = q_n(x) - w (> 0), \text{ and conversely.}$$

(d) Region IV (No production):

$$(y_n^*(x, w), z_n^*(x, w)) = 0$$

iff either

$$x < \bar{x}_n \text{ and } w \geq r_n(x)$$

or

$$x > \bar{x}_n \text{ and } w \geq q_n(x).$$

Proof: The proof is by induction on the period index n . To establish the initial case, $n=1$, one makes use of the special simple structure of B_1 resulting from the initial conditions along with assumption (f). The steps of this proof are exactly the same as those used to prove the general induction step. To conserve space, we provide only the proof of the general inductive step. Thus, it is assumed the theorem is true for periods $1, 2, \dots, n-1$ and show it true for period n .

First, make the following definitions:

$$\eta_1 y_n^u(x, w) = \bar{x}_n - x$$

$$z_n^u(x, w) = w_n(x) - w.$$

The proof of this theorem amounts to showing that the above definitions are valid and then showing that the conditions for each region as set forth

in Theorem 2 are met as stated in the proposition.

The defining conditions for (y_n^u, z_n^u) and (\bar{x}_n, \bar{w}_n) and the strict convexity of B_n assert the validity of the definitions above.

(a) If $x < \bar{x}_n$ and $w < w_n(x)$, then $(y_n^*, z_n^*) > 0$ as a consequence of Theorem 2. On the contrary, $y_n^* > 0$ implies that $x < \bar{x}_n$, which along with $z_n^* > 0$ implies that $w < w_n(x)$ as desired.

(b) If $x < \bar{x}_n$ and $w_n(x) < w$, it follows that $y_n^u > 0$, $z_n^u \leq 0$. Then, $w < r_n(x)$ implies that $B_n^{(1)}(0, 0, x, w) < 0$ which along with the strict convexity of B_n implies that $\bar{y}_n > 0$. Hence, $y_n^* > 0$, $z_n^* = 0$ as a consequence of Theorem 2. On the contrary, $y_n^* > 0$ and $z_n^* = 0$ imply that $y_n^u > 0$, $\bar{y}_n > 0$, $z_n^u \leq 0$ via Theorem 2. Now, $y_n^u > 0$ implies that $x < \bar{x}_n$ which together with $z_n^u \leq 0$ implies that $w > w_n(x)$. $x < \bar{x}_n$ and $\bar{y}_n > 0$ together with the strict convexity of B_n imply that $B_n^{(1)}(0, 0, x, w) < 0$ or simply $w < r_n(x)$. Finally, let $x^0 = x + \eta_1 \bar{y}_n$. Then

$$c + \eta_1 J_n^{(1)}(x^0, r_n(x^0)) + \eta_2 J_n^{(2)}(x^0, r_n(x^0)) = 0.$$

But, it also follows that

$$c + \eta_1 J_n^{(1)}(\eta_1 y_n^* + x, \eta_2 y_n^* + w) + \eta_2 J_n^{(2)}(\eta_1 y_n^* + x, \eta_2 y_n^* + w) = 0.$$

Therefore, the conclusion is that $\eta_2 y_n^* + w = r_n(x^0)$.

(c) If $x > \bar{x}_n$ and $w < q_n(x)$, then $y_n^u \leq 0$, and $\bar{z}_n > 0$ by the strict convexity of B_n . Theorem 2 is then applied to assert that $y_n^* = 0$ and $z_n^* > 0$. On the contrary, $y_n^u \leq 0$ and $\bar{z}_n > 0$ together imply that $x > \bar{x}_n$ and $w < q_n(x)$ by the strict convexity of B_n . Finally, it must be the case that

$$k + J_n^{(2)}(x, \bar{z}_n + w) = 0$$

and that

$$k + J_n^{(2)}(x, q_n(x)) = 0.$$

Therefore, it follows that $q_n(x) = \bar{z}_n + w$.

Q.E.D.

The reader is once again referred to Figure 2 as a visual help to understanding Theorem 4. Although Theorem 2 proves that there are four decision regions, it is this theorem that provides the insight into the model and the behavior of the optimal policy. Specifically, once the functions $r_n(\cdot)$, $q_n(\cdot)$ and $w_n(\cdot)$ and the points (\bar{x}_n, \bar{w}_n) have been generated with regard to one dimensional optimization, the following scheme should be followed in order to implement the policy in each period:

STEP 1 (Region I):

If $x > \bar{x}_n$ go to step 3. If not, then check to see if $w \geq w_n(x)$. If so, go to step 2. Otherwise, both A and B are to be used at optimal levels $y_n^*(x, w) = \eta_1^{-1}[\bar{x}_n - x]$ and $z_n^*(x, w) = w_n(x) - w$. Go to step 4.

STEP 2 (Region II): Set $z_n^*(x, w) = 0$.

If $w > r_n(x)$, do not produce at all this period. In this case, go to step 4. Otherwise, only type A is used with y_n^* solving

$$w + \eta_2 y_n^*(x, w) = r_n(x + \eta_1 y_n^*(x, w)).$$

Now, go to step 4.

STEP 3 (Region III): Set $y_n^*(x, w) = 0$.

If $w > q_n(x)$, do not produce at all this period. In this case, go to

step 4. Otherwise, type B will be used at level $z_n^*(x, w) = q_n(x) - w$.
Now, go to step 4.

STEP 4

If $n < N$, set $n \rightarrow n + 1$ go to step 1. Otherwise stop.

Summary

This article has presented a new class of multiproduct inventory systems, where decisions must be made at regular intervals of time (periods) concerning the optimal stock levels to maintain in order to anticipate stochastic demands. The novel feature of this new class is the notion of coordinating different types of production processes, operated in parallel (and independently), that are used to supply the stock items to the inventory system. Theoretical results were presented that characterized the optimal production policy in each period, in terms of functions $r_n(x)$, $q_n(x)$, and $w_n(x)$, and a point (\bar{x}_n, \bar{w}_n) . Various properties of these functions were also delineated, and it was shown that (\bar{x}_n, \bar{w}_n) is not necessarily the global minimum of the expected cost function. An implementation scheme was stated that provides guidelines for using the above results. The model explicitly assumed stationary demands and complete backlogging. The extension to nonstationary demands is quite straight forward, and is accomplished by adding superscripts to the densities, $f_j^n(\cdot)$, $j=1,2$ in all of the lemmas and theorems and also in Theorem 4.

The lost sales case can also be handled, in fact the form of the optimal policy remains the same. However, Theorem 2 must be proven in a slightly different manner. The details are described in Deuermeyer [4].

APPENDIX

This appendix contains the proofs of the theorems and lemmas needed to establish Theorem 4 stated in Section 3. The following well known result (see Rockafeller [12]) is used to prove Theorem 1.

Lemma 4: Let $C(x) = \min_{y \in D(x)} B(x, y) = B(x, y^*(x))$

where $y^*(x) \in D(x)$ and $x \in A$. Assume C and B are real valued. Then, if A is a convex set, B is jointly convex and if $X = \{(x, y); y \in D(x)\}$ is convex, then $C(x)$ is convex in x .

Proof of Theorem 1: The proof is by induction on n . First, consider $n=1$. The differentiability and strict convexity of B_1 are inherited from L .

(2) Let $\mathcal{E}_{xw}^1(\zeta) = \{(y, z); B_1(y, z, x, w) \leq \zeta\}$

where ζ is finite. Since B_1 is continuous it follows that $\mathcal{E}_{xw}^1(\zeta)$ is closed for each ζ . It suffices to show that $\mathcal{E}_{xw}^1(\zeta^0)$ is bounded for some arbitrary ζ^0 , since Rockafeller [12] has shown that if any level set of a closed convex function is bounded then they all are. Suppose $\mathcal{E}_{xw}^1(\zeta^0)$ is not bounded. Then, $(y^0, z^0) \in \mathcal{E}_{xw}^1(\zeta^0)$ and an unbounded sequence $\{(y_j, z_j)\}_j$ such that

$$\lim_{j \rightarrow \infty} B_1(y_j + y^0, z_j + z^0, x, w) \leq \zeta^0.$$

However, this statement contradicts Assumption f. Thus, $\mathcal{E}_{xw}^1(\zeta^0)$ is compact. The existence parts of (3) and (5) follow using the convexity and the compactness of $\mathcal{E}_{xw}^1(\zeta)$ with $\zeta = B_1(0, 0, x, w)$ and the Weierstrass theorem. The rest of (3) and (5) are merely statements of the first order conditions

for differentiable convex functions. Lemma 4 can be applied to prove (4) because B_1 is convex and the optimal policy is finite.

To complete the inductive argument it is assumed the theorem is true for $n-1$ and show it is true for period n . Part (1) is a consequence of the inductive hypothesis and the strict convexity of L . To prove (2) the continuity of B_n is used to assert the closedness of each level set

$$F_{xw}^n(\zeta) = \{(y,z); B_n(y,z,x,w) \leq \zeta\}$$

for every $\zeta \in [0, \infty)$. It remains to show that for any fixed finite $\zeta^0 \in [0, \infty)$ that $F_{xw}^n(\zeta^0)$ is bounded. Suppose not; then $E(y^0, z^0)$ and an unbounded sequence $\{(y_i, z_i)\}_i$ such that

$$(1) \quad \lim_{i \rightarrow \infty} B_n(y_i + y^0, z_i + z^0, x, w) \leq \zeta^0.$$

But, since $C_{n-1}(x, w) \geq 0$ for all (x, w) , it follows that

$$B_n(y, z, x, w) \geq B_1(y, z, x, w).$$

It then follows that (1) can not hold because B_1 is unbounded. The rest of the proof repeats the same steps as in the $n=1$ case, by simply replacing 1 by n .

Q.E.D.

Proof of Theorem 2: The proof is by induction on n , the period index. The proof relies on the Kuhn-Tucker saddle-point theory for convex functions, and relies on the information supplied by Theorem 1, namely the strict convexity of B_n and the finiteness of the optimal policy. Since the proof of the $n=1$ case is exactly the same as the general inductive step, we provide only the general step. By substituting 1 for n in the proof to follow and

making use of the initial conditions previously stated, the proof of the $n-1$ case is immediate. Thus, assume the theorem is true for periods $1, 2, \dots, n-1$ and we now show it is true for period n .

For an unconstrained problem such as is the case here (except for the nonnegativity) the optimal policy must solve the Kuhn-Tucker saddle point conditions as put forth in Mangasarian [11]. In the present context, it follows that $(y_n^*(x, w), z_n^*(x, w))$ is the optimal policy iff this point solves

$$\begin{aligned}
 (K1) \quad & B_n^{(1)}(y_n^*(x, w), z_n^*(x, w), x, w) \geq 0 \\
 (K2) \quad & B_n^{(2)}(y_n^*(x, w), z_n^*(x, w), x, w) \geq 0 \\
 (K3) \quad & y_n^*(x, w) \cdot B_n^{(1)}(y_n^*(x, w), z_n^*(x, w), x, w) = 0 \\
 (K4) \quad & z_n^*(x, w) \cdot B_n^{(2)}(y_n^*(x, w), z_n^*(x, w), x, w) = 0 \\
 (K5) \quad & (y_n^*(x, w), z_n^*(x, w)) \geq 0
 \end{aligned}$$

The theorem is proved as outlined below, using (K1) - (K5).

(a) If $(y_n^*(x, w), z_n^*(x, w)) > 0$, then $B_n^{(1)}(y_n^*(x, w), z_n^*(x, w), x, w) = 0$, $i=1, 2$ by (K3) and (K4). In this case, $(y_n^*(x, w), z_n^*(x, w))$ solves the same system that uniquely defines $(y_n^u(x, w), z_n^u(x, w))$, so these points are identical. On the other hand, if $(y_n^u(x, w), z_n^u(x, w)) > 0$, it is easy to show it solves (K1) - (K5) establishing the desired equivalence.

(b) The proofs of (b) and (c) are essentially the same so only (b) is

shown here. For simplicity, the abbreviated notation $y_n^* \equiv y_n^*(x, w)$, $z_n^* \equiv z_n^*(x, w)$, etc., will be used. Now, $y_n^* > 0$ and $z_n^* = 0$ imply $y_n^u > 0$ and $z_n^u \leq 0$. (K1) and (K3) imply that

$$B_n^{(1)}(y_n^*, 0, x, w) = B_n^{(1)}(y_n^*, z_n^*, x, w) = 0$$

or that $y_n^* = \bar{y}_n$ so $\bar{y}_n > 0$ as well. Hence, $y_n^* > 0$, $z_n^* = 0$ implies $y_n^u > 0$, $\bar{y}_n > 0$, and $z_n^u \leq 0$.

To prove the converse note that

$$y_n^u > 0 \text{ and } z_n^u \leq 0 \text{ imply } y_n^* \geq 0 \text{ and } z_n^* = 0.$$

Therefore,

$$B_n^{(1)}(y_n^*, 0, x, w) = B_n^{(1)}(y_n^*, z_n^*, x, w).$$

Since $(y_n^*, 0)$ necessarily solves (K1) - (K5), it must solve $B_n^{(1)}(y_n^*, 0, x, w) \geq 0$ and

$$y_n^* \cdot B_n^{(1)}(y_n^*, 0, x, w) = 0.$$

But, \bar{y}_n solves $B_n^{(1)}(\bar{y}_n, 0, x, w) = 0$ so $(\bar{y}_n, 0)$ will also solve (K1) - (K5). The uniqueness of \bar{y}_n with this argument asserts that $\bar{y}_n = y_n^* > 0$. Therefore,

$$y_n^u > 0, z_n^u \leq 0 \text{ and } \bar{y}_n > 0 \text{ implies } y_n^* > 0 \text{ and } z_n^* = 0.$$

(d) The validity of this part follows directly from the contrapositives of parts (a) - (c). For example, if (i), (ii), or (iii) hold, then $(y_n^*, z_n^*) = 0$ by the contrapositive of (a), (b) or (c), respectively. Conversely, if $(y_n^*, z_n^*) = 0$, one of (i) - (iii) must hold.

Q.E.D.

Proof of Theorem 3: The $n=1$ case that initiates the inductive argument fol-

lows in similar fashion to the general step and is omitted only to conserve space. Assume the theorem is true for periods $1, 2, \dots, n-1$; it is to be shown true for period n .

(1) $C_{n-1}^{(j)}$ is a piecewise differentiable function by the inductive hypothesis applied to the final conclusion for $j=1, 2$. Therefore, the expected value of $C_{n-1}^{(j)}$ is continuously differentiable by the continuity of $f_j(\cdot)$. From this, it follows that B_n has continuous second partial derivatives.

(2) Recall that

$$B_n(y, z, x, w) = cy + kz + J_n(\eta_1 y + x, z + \eta_2 y + w).$$

Therefore, in region I, $y_n^* = y_n^*(x, w)$ and $z_n^* = z_n^*(x, w)$ solves

$$c + \eta_1 J_n^{(1)}(y_n^*, z_n^*, x, w) - \eta_2 k = 0$$

$$k + J_n^{(2)}(y_n^*, z_n^*, x, w) = 0.$$

Since B_n is strictly convex the Implicit Function Theorem can be applied to assert the differentiability of the optimal policy in an open neighborhood of (x, w) in the interior of region I; moreover the partials are given by

$$y_n^{*(1)}(x, w) = -\frac{1}{A} \begin{vmatrix} \eta_1 J_n^{(1,1)} & \eta_1 J_n^{(2,1)} \\ J_n^{(1,2)} & J_n^{(2,2)} \end{vmatrix} = -\eta_1^{-1}$$

$$y_n^{*(2)}(x, w) = -\frac{1}{A} \begin{vmatrix} \eta_1 J_n^{(1,2)} & \eta_1 J_n^{(1,2)} \\ J_n^{(2,2)} & J_n^{(2,2)} \end{vmatrix} = 0$$

$$z_n^{*(1)}(x,w) = -\frac{1}{A} \begin{vmatrix} \eta_1^2 J_n^{(1,1)} + \eta_1 \eta_2 J_n^{(1,2)} & \eta_1 J_n^{(1,1)} \\ \eta_1 J_n^{(1,2)} + \eta_2 J_n^{(2,2)} & J_n^{(2,2)} \end{vmatrix} = \eta_2 \eta_1^{-1}$$

$$z_n^{*(2)}(x,w) = -\frac{1}{A} \begin{vmatrix} \eta_1^2 J_n^{(1,1)} + \eta_1 \eta_2 J_n^{(2,1)} & \eta_1 J_n^{(2,1)} \\ \eta_1 J_n^{(1,2)} + \eta_2 J_n^{(2,2)} & J_n^{(2,2)} \end{vmatrix} = -1$$

where

$$A = \begin{vmatrix} \eta_1^2 J_n^{(1,1)} + \eta_1 \eta_2 J_n^{(1,2)} & \eta_1 J_n^{(2,1)} \\ \eta_1 J_n^{(1,2)} + \eta_2 J_n^{(2,2)} & J_n^{(2,1)} \end{vmatrix}$$

and $J_n^{(i,j)} = J_n^{(i,j)}(\eta_1 y_n^* + x, z_n^* + \eta_2 y_n^* + w)$, $i, j = 1, 2$.

The remainder of the proof proceeds in much the same fashion using the defining identities of the optimal policy in each of the remaining regions.

In addition, the following relations must be verified:

$$J_n^{(i,i)}(x,w) > 0, \quad i=1,2$$

$$J_n^{(1,2)}(x,w) = J_n^{(2,1)}(x,w) \leq 0$$

$$\eta_1 J_n^{(j,1)}(x,w) + \eta_2 J_n^{(j,2)}(x,w) > 0, \quad j=1,2.$$

(3) The finiteness of (y_n^*, z_n^*) implies that

$$C_n(x,w) = cy_n^* + kz_n^* + J_n(\eta_1 y_n^* + x, z_n^* + \eta_2 y_n^* + w).$$

The proof of this part is very straight forward and amounts to differentiating the above relation for (x,w) in each of the individual regions. The remainder of the proof is omitted.

Q.E.D.

Proof of Lemma 3: The proof is by induction on the period index. The $n=1$ case follows exactly the same proof of the general n step making use of the initial conditions previously stated, with n set to 1. The explicit proof is omitted simply to conserve space. It is assumed the lemma is true for periods $1, 2, \dots, n-1$ and is to be shown valid for period n . Consider the strictly convex function

$$Q_n(x, w) = B_n(0, 0, x, w).$$

(1) By definition, (\bar{x}_n, \bar{w}_n) solves

$$Q_n^{(i)}(\bar{x}_n, \bar{w}_n) = B_n^{(i)}(0, 0, x, w) = 0, \quad i=1, 2.$$

which is finite and unique.

(2) The strict convexity of Q_n and the Implicit Function Theorem assert the finiteness, uniqueness, and differentiability of the point $r_n(x)$ that solves

$$Q_n^{(1)}(x, r_n(x)) = B_n^{(1)}(0, 0, x, r_n(x)) = 0$$

for $x \leq \bar{x}_n$.

(3) The function $q_n(x)$ is finite and unique using a similar argument to that in (2) above.

(4) The uniqueness and finiteness together imply that $w_n(x) = \bar{w}_n - \eta_2 \eta_1^{-1}(\bar{x}_n - x)$ is well defined.

(5) (a) The Implicit Function Theorem and the strict convexity of B_n imply that

$$r_n'(x) = - \frac{[\eta_1 J_n^{(1,1)}(x, r_n(x)) + \eta_2 J_n^{(1,2)}(x, r_n(x))]}{[\eta_1 J_n^{(2,1)}(x, r_n(x)) + \eta_2 J_n^{(2,2)}(x, r_n(x))]} < 0.$$

In addition, it follows that $r_n(\bar{x}_n) = \bar{w}_n$ because

$$B_n^{(1)}(0, 0, \bar{x}_n, r_n(\bar{x}_n)) = 0 = B_n^{(1)}(0, 0, \bar{x}_n, \bar{w}_n)$$

and both $r_n(\bar{x}_n)$ and \bar{w}_n are uniquely defined.

(b) In similar fashion to (a) it follows that

$$q_n'(x) = - \frac{J_n^{(1,2)}(x, q_n(x))}{J_n^{(2,2)}(x, q_n(x))}.$$

The proof of Theorem 3 established the two inequalities

$$\begin{aligned} J_n^{(1,2)}(x, w) &\leq 0 \\ J_n^{(1,2)}(x, w) + J_n^{(2,2)}(x, w) &\geq 0. \end{aligned}$$

Therefore, it follows that $0 \leq q_n'(x) \leq 1$ provided $x \leq \bar{x}_n$. Finally,

$$B_n^{(2)}(0, 0, \bar{x}_n, q_n(\bar{x}_n)) = 0 = B_n^{(2)}(0, 0, \bar{x}_n, \bar{w}_n)$$

which implies that $q_n(\bar{x}_n) = \bar{w}_n$.

(c) This part is obvious.

Q.E.D.

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report Number 4	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A By-Product Production Process with an Alternative for a Two-Product Inventory System		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER 14TR-4
7. AUTHOR(s) Bryan L. Deuermeyer William P. Pierskalla		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0797
9. PERFORMING ORGANIZATION NAME AND ADDRESS Dept. of Industrial Engineering and Management Northwestern University Evanston, Illinois 60201		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042-322
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics and Probability Program Code 436 Arlington, Virginia 22217		12. REPORT DATE December, 1976
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES 121
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES 402 526		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Production Process, By-Products, Inventory Control		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper considers the optimal control of a production system which is composed of two distinct production processes, called type A and type B. These production processes are used to produce two different products called I and II. It is assumed that the demands for these products are sequences of continuous, nonnegative and independent random variables with known distributions. Production type A is used to produce both products, but they are related as by-products of each other by a fixed set of production coefficients, while type B		

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cont → can only be used to produce product II. At the beginning of each period, the level of production of each production type must be determined, given the amounts of each product already in stock. The criterion used to make these decisions is based on the minimum expected discounted total cost over the finite horizon. Results presented show that the decision plane is partitioned into four distinct regions, and the boundaries between these regions are characterized in terms of a particular sequence of points which are independent of the on-hand stock levels each period. ↑

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